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Observations on a spatial-resonance phenomenon

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The phenomenon was observed during experiments in which a beaker containing water was vibrated in one of its bell modes (the inextensional flexural vibrations of the wall). For certain combinations of driving force and frequency, a standing water wave of large amplitude was generated whose peripheral wavenumber might be either zero (i.e. the wave was radially symmetric) or twice that of the bell mode. This relationship between the wavenumbers of the bell mode and water wave, and the fact that the driving frequency was many times that of the water wave, indicated that this was an instance of a general mechanism that has been studied theoretically by Mahony & Smith (1972). For a model situation, allowing for dissipative effects and nonlinear coupling between nearly resonant oscillations at greatly differing frequencies, they derived a relationship between the driving force and frequency representing conditions of neutral stability (i.e. such that the rate of energy transfer from the high frequency to the low frequency oscillations is zero). The aim of the experimental observations reported here was to check this relationship and other predictions of their theory.

1. Introduction

The phenomenon under study was first noticed in the course of other experiments on liquids subject to high frequency vibrations. A beaker containing water was vibrated continuously near resonance in one of its bell modes,† and in certain ranges of the frequency and amplitude of these vibrations standing waves were observed to build up in the water, generally reaching amplitudes very much larger than that of the elastic vibrations of the beaker wall. The outstanding and initially puzzling feature of the phenomenon was that the water wave frequency was a very small fraction (typically about $\frac{1}{50}$) of the excitation frequency, so that none of the familiar mechanisms of energy transfer between wave modes, for example, 'parametric' excitation modelled by the unstable solutions of Mathieu's equation, appeared to be relevant. Another remarkable feature was that if the bell mode had a peripheral wavenumber $k \geqslant 2$ (i.e. there were $2k$ nodes spaced around the circumference of the beaker) then water waves might be generated with a wavenumber of either zero or $2k$.

In fact both these features indicate that the phenomenon is a good example of a general mechanism recently discovered by Mahony & Smith (1972), whose

† These are the gravest elastic vibrations of an open-ended shell: the modes in which a wine glass rings after being tapped.

theoretical paper on the subject appears at the same time as this paper. The mechanism described by them provides for the transfer of energy, through nonlinear coupling, between lightly damped nearly resonant oscillations at greatly differing frequencies, and is such that the overall system can be in an unstable state when its energy is concentrated in the high frequency mode. Mahony & Smith introduced the term spatial resonance for the general phenomenon; another likely instance of it is also mentioned in their paper.†

The following theoretical results obtained by them have been tested by the present experiments. First, we recall the familiar response relationship, as used in their theory, for the forced simple harmonic oscillations of a resonant system with linear damping. Thus, adopting most of their notation, we have for the high frequency elastic vibrations

$$A = B/(\Delta^2+\nu^2)^{\frac{1}{2}}, \tag{1}$$

where A is the amplitude of the displacements of the beaker wall (at any representative point, not a node), B is proportional to the amplitude of the driving force applied to the beaker, ν is the logarithmic decay rate of free vibrations and $\Delta = \omega-\omega_0$ is the difference between the (radian) frequency ω of the excitation and the resonance frequency ω_0. The particular results to be tested (equations (17a, b) in Mahony & Smith's paper) express conditions of neutral stability; they were originally given in terms of the excitation amplitude B, but we now use the formula (1) to express them rather more conveniently for present purposes. Thus the critical value A_c of the vibration amplitude A, above which the high frequency vibrations become unstable and lose energy to the water waves, is given by

$$A_c^2 = \begin{cases} \dfrac{\sigma(\Delta^2+\nu^2)}{2\gamma|\Delta|} \quad \text{if} \quad \Delta < 0, & (2a) \\[2ex] \dfrac{\nu'\{(\Delta^2+\nu^2-\sigma^2)^2+4\nu^2\sigma^2\}}{4\nu\sigma\gamma\Delta} \quad \text{if} \quad \Delta > 0, & (2b) \end{cases}$$

where σ is the water wave frequency, ν' is the damping rate for free water waves and γ ($=\alpha\beta$ in Mahony & Smith's notation) is a parameter of the physical system relating to the nonlinear coupling between the high and low frequency modes. It was assumed by Mahony & Smith that γ is real and positive and this assumption will be checked in §3.

With the equipment to be described, the frequencies ω, ω_0, σ and the damping rate ν' could all be measured directly. Also, A could be measured as a function of ω for constant B, and hence by comparison with the theoretical relation (1) the decay rate ν for the high frequency vibrations could be estimated. The curve of neutral stability was obtained by plotting A_c against Δ. Each point on the curve was obtained by plotting observed growth rates against A and then extrapolating to zero growth rate. The extrapolation was guided by the theoretical prediction (implied by equation (16) in Mahony & Smith's paper) that at a fixed ω the growth rate is proportional to $A^2-A_c^2$. Using the values of ν, ν' and σ

† See also Lindholm, Kana & Abramson (1962).

obtained experimentally, it was then possible to plot the curve of neutral stability as given by equations (2) for comparison with the experimental results.

2. Experimental apparatus and procedure

The rim was removed from a 4 litre Pyrex beaker and the beaker was then glued onto a rubber sheet of $\frac{1}{4}$ in. thickness fastened to a wooden board. The electro-magnetic transducer used to drive the beaker and the probe of the metering system described below were also fixed rigidly to this horizontal board. The driving spindle of the transducer was glued to a point about mid-way between the water level and the base.

A 'Wayne Kerr' vibration/distance meter with its output displayed on an oscilloscope was used to measure the displacement of the wall of the vibrating beaker. This meter is sensitive to the electrical capacitance between a stationary probe and a moving conducting surface which has to be grounded, so to employ it for the present purpose a strip of silver paper was taped to the wall of the beaker and grounded by a fine wire. The transducer ('Pye–Ling' model 101) was driven by the amplified signal from a stabilized oscillator, and an electronic counter was used to record the working frequency, which could be set to within 0·05 Hz.

Although it is often possible to use such a proximity meter directly to measure the displacements of a water surface, this method proved impracticable for the water waves in most of the present experiments, owing to the presence of vigorous capillary waves and spray formation at the antinodes of the vibration of the beaker walls. [These capillary waves were examples of the 'crispations' first studied by Faraday, as described by Rayleigh (1894, §354).] Accordingly, the low frequency standing waves excited in the water were monitored by observing the modulations in the response of the beaker walls. This method proved sensitive enough to enable the growth-rate measurements to be made before the waves had grown to visible size. The proximity meter was used, however, to measure the decay rate for the (free) water waves; a capacitive probe similar to but larger than the one used to measure the beaker vibrations was then mounted above the free surface of the water, which was grounded by an immersed wire.

In both the aforementioned applications of the proximity meter to measure displacements, manufacturer's tables indicated that no correction was needed for the effects of curvature of the moving surface.

2.1. *Response curve*

To obtain the response curve for the primary vibrations of the beaker, the transducer was driven with a constant current (of about 0·1 amps r.m.s. into the 3 ohm transducer) while the frequency was varied. The output from the proximity meter was noted from the oscilloscope trace.

2.2. *Growth rates*

It proved essential to keep the water clean, especially the surface, in order to obtain consistent results. Distilled water was used, with a crystal of thymol dissolved in it to inhibit bacteriological effects, and a vacuum line from a jet pump was used to skim the surface before each experimental run. The water was then topped up to a standard level. It was found that small changes (up to about 0·2 %) were liable to occur in the resonance frequency of the system, so Δ was evaluated from the resonance frequency observed during the experimental run in question.

To measure the growth rate of the water waves at a particular frequency the transducer was driven with a constant current and a record was taken of the times at which the amplitude of modulation on the high frequency vibration passed a succession of predetermined values. When conditions were near the threshold of stability, however, the time scale for appreciable growth was inconveniently long, of the order of several minutes. In this case the surface was given a small disturbance artificially and the growth or decay of the resulting waves was noted. This procedure greatly aided the extrapolation to zero growth rate on a plot of the growth rate against A_c^2.

2.3. *Determination of ν' and σ*

The decay rate ν' for free-surface waves was determined in a manner very similar to that used to obtain the growth rates arising from the instability mechanism. Water waves were set up by the latter mechanism and then the high frequency drive was switched off. With the capacitive probe mounted above the water surface, the amplitude of the oscilloscope trace was noted as a function of time. The period $2\pi/\sigma$ of the water waves was read off from the trace at the same time, as a check on a measurement made by stop-watch.

3. Experimental results and discussion

Figure 1 shows the experimental response curve and model curve giving the best fit, for which the chosen value of ν is 0·95 s^{-1}. The decay curve for the radially symmetric water wave is shown in figure 2. From this the experimental value of ν' is estimated to be 0·028 s^{-1}.

In figure 3 a typical set of growth-rate measurements at fixed frequency is shown, from which the value of A_c^2 can be obtained for the particular frequency.

There is no obvious reason why γ should be real for the experimental system used. So Mahony & Smith's calculation of neutral-stability conditions was repeated with allowance for complex values of γ. This gave an expression analogous to (2b), and a least-squares fit with the experimental data gave the imaginary part of γ to be only about $\frac{1}{1000}$ of the real part, so it can justifiably be ignored as this is well within the margin of experimental error. Reverting to Mahony & Smith's expression (17), it is now possible to calculate the minimum value of A_c according to our equation (2a). With γ taken to be real this turns out to be

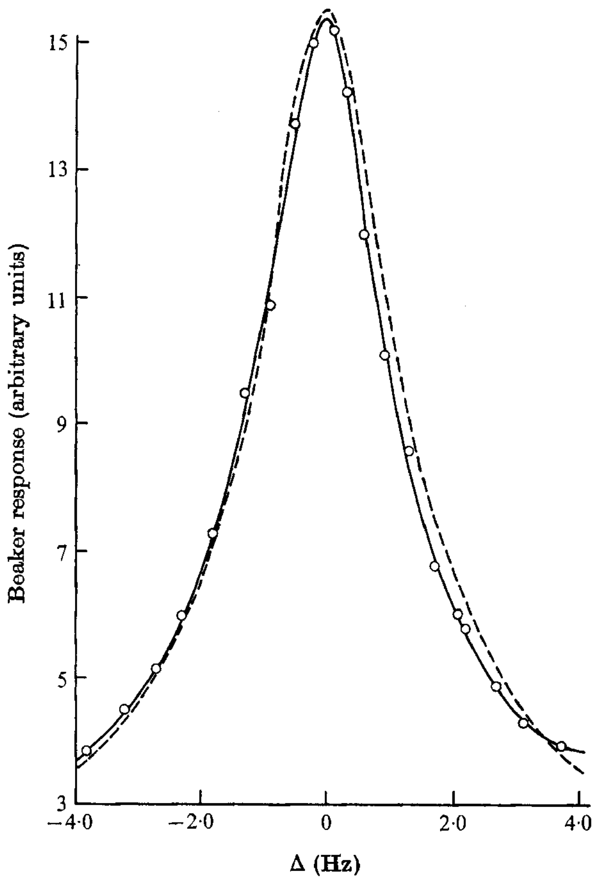

FIGURE 1. Theoretical and experimental response curves. Δ is the difference between the working frequency and the resonant frequency. - - -, theory; ——, experiment.

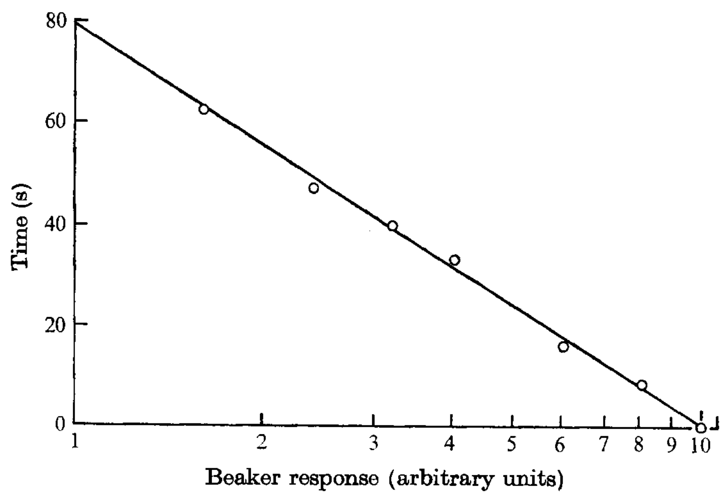

FIGURE 2. Decay of the radially symmetric water wave.

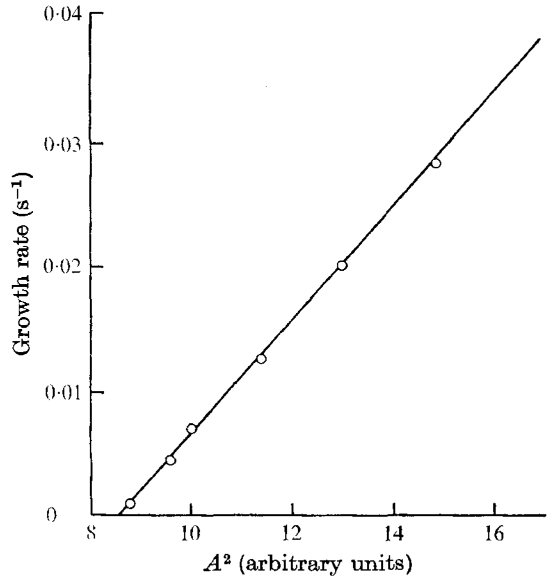

FIGURE 3. Growth rate against (response)^2 for fixed frequency.

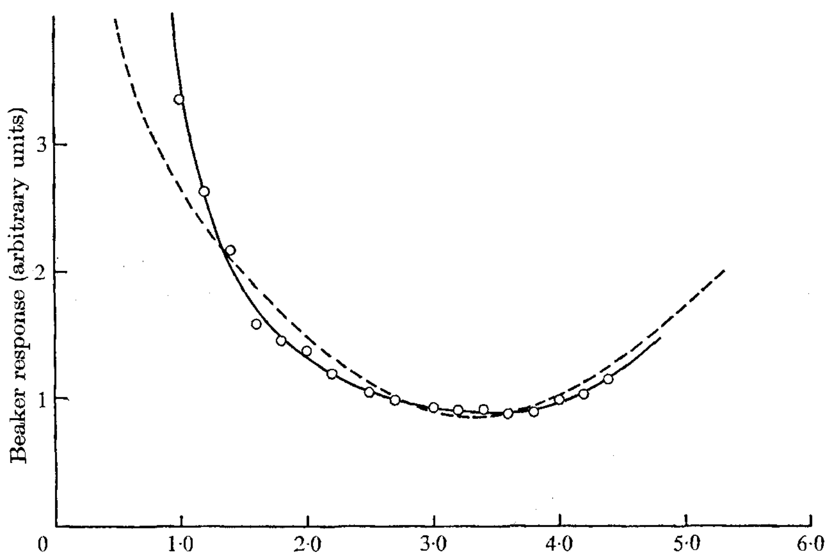

FIGURE 4. Theoretical and experimental curves showing the threshold of stability. – – –, theory; ——, experiment.

well beyond the capabilities of the transducer used, and so explains why no surface waves could be generated for $\Delta < 0$.

Figure 4 gives the theoretical and experimental curves of neutral stability for $\Delta > 0$, with respect to the symmetric water wave mode. It was not possible to extend the experimental curve beyond $\Delta = 4{\cdot}4$ Hz, owing to the presence of the waves with peripheral wavenumber 4. These waves were anticipated

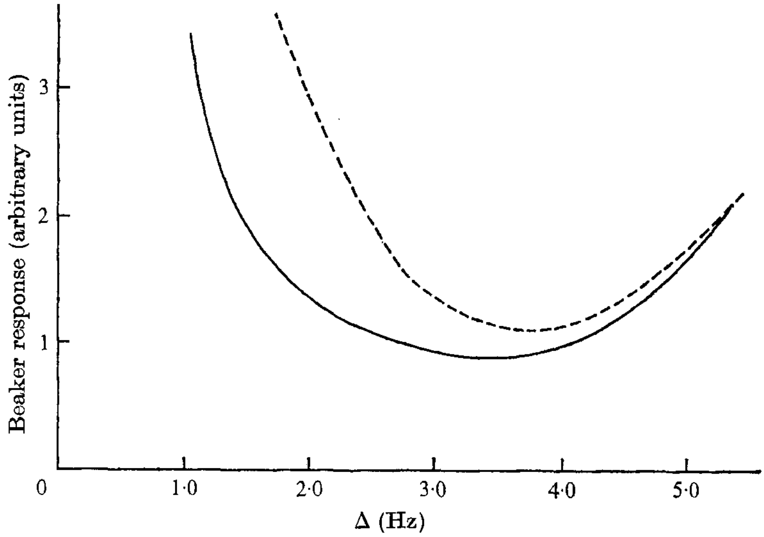

FIGURE 5. Suggested relationship between the curves of neutral stability for two types of water wave. – – –, wave with peripheral wavenumber 4; ——, radially symmetric wave.

from the model calculation of Mahony & Smith, but it proved impossible to sustain them for long enough to make any accurate measurements; they always interacted with and eventually became dominated by the radially symmetric waves. For some values of Δ, however, this interaction proceeded very slowly. The motion then observed was approximately radially symmetric, but the predominant frequency was 5 Hz, in contrast with the resonance frequency $\sigma = 3\frac{1}{3}$ Hz for the free symmetric waves. Since the waves with wavenumber 4 had a frequency of about 3·4 Hz, this appears to be a high order interaction beyond the scope of the model calculation. It was possible to excite the interaction when Δ lay in the range 4·4–5·0 Hz but for larger values of Δ there was insufficient power available from the transducer. It ought to be mentioned also that the waves with wavenumber 4 have been generated for values of Δ down to 2·6 Hz, but not below this figure. From this, and the fact that the minimum value of A_c given by equation (2*b*) occurs for

$$\Delta^2 = \tfrac{2}{3}(\nu^4+\nu^2\sigma^2+\sigma^4)^{\frac{1}{2}} - \tfrac{1}{3}(\nu^2-\sigma^2)$$

(which is monotonic increasing with σ), it appears that the relationship between the neutral-stability curves for the two different water wave modes is as shown in figure 5. The two curves are so close to each other for $\Delta > 4·4$ Hz that one cannot expect to generate either wave without the other.

The agreement between the experimental results and Mahony & Smith's model calculation is considered to be quite satisfactory, justifying reasonable confidence in the validity of their theory. The discrepancy for small Δ is probably due largely to the presence of the Faraday capillary waves, which are an additional dissipative mechanism for large driving amplitudes.

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